9 Algebra and equations

1 Variables

A variable is a symbol that represents a number. We usually use letters such as x, n, p, t for variables.

Letters are useful if we want to operate with an unknown number instead with a particular one. Let us look at some examples:

We say that *s* represents the side of a square, then *s* represents a number and:

$4s$ Is the perimeter of the square

$s \cdot s$ Is the area of the square

When letters express numbers they have the same operating properties. The part of mathematics that deals with the study of the expressions with letters and numbers is called **algebra**.

2 Expressions

An expression is a mathematical statement with numbers and variables.

Examples:

3 x $x+3$ $2\cdot(x-5)$ x^2-3x

If Mark weights 80 kg and he gains n kg, the new weight is 80 + n

Exercise 1 Calling '*a*' the age of a person write an expression for:

1.1 The age he/she will be in 2012.

1.2 The age he/she was 7 years ago

1.3 The age he/she will be after living the same time again.

Exercise 2 Calling x a number, express in algebra:

2.1 The sum of the number and 10

2.2 The difference between 123 and the number

2.3 The double of the number

2.4 The triple of the number plus three units

2.5 The half of the number minus seven

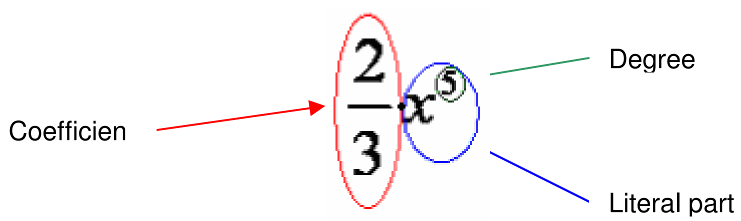
2.6 The three quarters of the number plus forty-six

3 Monomials

The simplest algebraic expressions formed by products of numbers and letters are called monomial.

A monomial consists of the product of a known number (coefficient) by one or several letters with exponents that must be constant and positive whole numbers (literal part).

Generally in the monomials the product signs are not included, so we find a number followed by one or more letters, we understand that they are multiplied.



Examples:

a) 2x is a monomial. 2 is the coefficient, x is literal part.

b) $-3x^2$ is a monomial, -3 is the coefficient, x^2 is the literal part, x is the variable and the degree is 2

c) $\frac{3}{5}t^7$ is a monomial, $\frac{3}{5}$ is the coefficient, t^7 is the literal part, t is the variable and the degree is 7

d) $5xy^2$ is a monomial, 5 is the coefficient, xy^2 is the literal part, x and y are the variables and the degree is 3

e) $2x + 7$ is an algebraic expression but it is not a monomial.

f) $\frac{3}{x}$ is an algebraic expression but it is not a monomial.

Exercise 3 Find which of the following expressions are monomials and determine, if they are so, their coefficient, literal part and variables:

a) $-\frac{1}{5}x^7$

b) $2t^2$

c) $a + b$

d) a^9

e) $\sqrt{2}n^4$

f) $3\sqrt{n}$

g) $7abc^2$

4 Operations of monomials

4.1 Addition or subtraction of two monomials.

We add or subtract the coefficients and we leave the literal part unchanged.

Two monomials can only be simplified when both have the same literal part, that is, when they are **“like monomials”**. When the literal part is different, the addition is left indicated.

Examples:

$5x + 7x = 12x$

$\frac{1}{2}x + 2x = \left(\frac{1}{2} + 2\right)x = \frac{5}{2}x$

$7ab + 2ab = 9ab$

Exercise 4 Operate

a) $2x - 7x =$ b) $15x + 2x =$ c) $8x - x =$

d) $\frac{15}{2}x + 2x =$ e) $-2x^2 + 23x^2 =$ f) $15x + 2 =$

g) $3a - 7a =$ h) $x^2 + 3x$ i) $x + \frac{2}{3}x =$

4.2 Product of two monomials.

We multiply the coefficients and also the literal part (remember how we multiply powers with the same base).

Examples:

$5x \cdot 7x = 35x^2$

$\frac{1}{2}x^2 \cdot 2x = x^3$

$7a^3 \cdot 2a^2 = 14a^5$

Exercise 5 Operate

a) $2x \cdot 7x =$ b) $15x \cdot 2x =$ c) $8x \cdot x =$

d) $\frac{15}{2}x \cdot 2x^3 =$ e) $-2x^2 \cdot 23x^2 =$ f) $15x \cdot 2 =$

5 Manipulating algebraic expressions

5.1 Evaluate an expression.

To evaluate an expression at some number means we replace a variable in the expression with the number and then, if necessary, we calculate the value.

Example:

Evaluate $5x+3$ when $x=2$, $5x+3$ becomes $5 \cdot 2+3=13$

Remember the rules: **parenthesis, exponents, multiplications and divisions and, the last, additions and subtractions.**

Exercise 6 Evaluate the expressions:

a) $-\frac{1}{5}x^2$ when $x=3$

b) $2t^2$ when $t=2$

c) a+b when $a=b=7$

d) a^9 when $a=-1$

e) n^4+3n when $n=3$

f) $\frac{3n+2}{n-3}$ when $n=5$

g) $7a + b - c^2$ when $a = 3, b = -2, c = 4$

5.2 Expansion of brackets.

Multiplying a number by an addition is equal to multiplying by each adding number and then to adding the partial products.

Examples:

$2(x + 6) = 2x + 12$

$3(2x-7) = 6x-21$

$3x(3x+2) = 9x^2+6x$

Exercise 7 Fill in the missing terms:

a) $5(3x-4) = \square - 20$ b) $x(x-2) = \square -2x$

c) $a(4-a) = \square - a^2$ d) $7x(3x-y) = \square - 7xy$

Exercise 8 Expand:

a) $2(x - 4) =$ b) $6(2 - x) =$

c) $6x(2x - 1) =$ d) $2t(2t + 8) =$

e) $5x(2x - 3y) =$ f) $x(x - 1) =$

g) $2(7 + 7x) =$ h) $2x(3x - a) =$

Exercise 9 Write down expressions for the area of these rectangles and then expand the brackets as in the example.

a) 3x+1

3 $A = 3(3x + 1) = 9x + 3$

b) 2x-1

7 A =

c) 3x

x-2 A =

d) x-7

4 A =

5.3 Common factors

It is important to know how to write expressions including brackets when it is possible; this is called **factorization** or **factoring**

Examples:

$$12x + 6 = 6 \cdot 2x + 6 \cdot 1 = 6(2x + 1)$$
$$= 6(2x + 1)$$

$$21x - 14 = 7 \cdot 3x - 7 \cdot 2 = 7(3x - 2)$$
$$= 7(3x - 2)$$

$5x^2 - 8x = x(5x - 8)$ We can check our answer by expanding the expression.

Exercise 10 Factorise:

a) $10x + 15 =$ b) $4x - 12 =$ c) $9x - 9 =$

d) $3x^2 - 2x =$ e) $9x^2 + 3x =$ f) $3x - 3 =$

g) $33x^2 - 3x =$ h) $5x - 3x^2 =$ g) $13x^2 - 2 =$

6 Equations

An equation is a statement in which two expressions are equal.

- The letter in an equation is called the **unknown**. (Sometimes it is called the **variable**).

Examples of equations:

$x = 2$ $3 = x$ $2x = 4$ $5x+1 = 3$ $x^2 = 4$

In an equation we consider two members:

First member is the left member
Second member is the right member.

Each monomial is called a term. Monomials with the same literal part are called **"like terms"**

Example:

In the equation $3t - 2 = -7$ we say:

- The first member is $3t - 2$
- The second member is -7
- There are three terms 3t, -2 and –7
- The unknown is t.

In the equation $3x^2 - 6 = 5x + 2$:

- The first member is $3x^2 - 6$
- The second member is $5x + 2$
- There are four terms $3x^2$, -6, 5x and 2
- The unknown is x.

7 Solving equations

Solving an equation is to find out a solution.
A solution to an equation is the number that makes the equality true when we replace the unknown or variable with that number.

Examples:

2 is a solution to the equation $2x = 4$ because if we replace x by 2 that gives us $2 \cdot 2 = 4$, which is true.

3 is a solution to the equation $5x + 8 = 16$, if we replace x by 3, we have $5 \cdot 3 + 8 = 16$, which is true.

4 is not a solution to $3x - 1 = 4$ because $3 \cdot 4 - 1 = 4$ is not true.

7.1 Rules of simplifying and solving equations

These are the basic rules for simplifying equations:

These rules must be followed, usually in this order, to solve linear equations.

Rule 1 If there are brackets we must remove them (**expand the expression**) as in the usual operations with numbers.

Rule 2 We can add or subtract any number to or from both members of the equation so **any x-term or number that is adding (positive sign) moves to the other side subtracting (negative sign) and vice versa.**

Rule 3 Move x-terms to one side and numbers to the other.

Rule 4 Combine like terms

Rule 5 Any number **that is multiplying,** the whole expression, on one side moves to the other side **dividing,** and a number that is **dividing,** the whole expression, on one side moves to the other **multiplying**.

Examples:

1. Solve $5x + 12 - 4 = -2$

We add the two numbers in the first member, we get

$$5x + 8 = -2$$

We may subtract 8 from each member and make the addition of the integers numbers that appear

$$5x + 8 - 8 = -2 - 8,$$

$$5x + 0 = -2 - 8, \; 5x = -2 - 8$$

More easily , 8 that is positive moves to the right member as -8

$5x = -10$

We divide each member by 5 or we can say that 5 which is multiplying moves to the second member dividing $x = \frac{-10}{5}$

$x = -2$ we can check the solution (always in the original equation)

$5 \cdot 2 + 12 - 4 = -2$

$10 + 12 - 4 = -2$

$-2 = -2$ So our solution is correct.

2. Solve $7x + 5 = 2x$

We may subtract 5 from each member and make the addition of the integers numbers that appear

$7x + 5 - 5 = 2x - 5$

$7x = 2x - 5$

We may subtract 2x from each member

$7x - 2x = 2x - 2x - 5$

We can add like terms

We divide each member by 5

$5x = -5$

We divide each member by 5 and we get

$x = -1$ We check the solution

$7 \cdot (-1) + 5 = 2(-1)$

$-7 + 5 = -2$

$-2 = -2$ So our solution is correct.

3. Solve $2(x + 5) + 1 = 17$

We can expand

$2x + 10 + 1 = 17$

$2x + 11 = 17$

We may subtract 11 from each member

$2x+11-11=17-11$ or simply 11 moves to the right member as -11

and then $2x=17-11$

$2x=6$

We divide each member by 2 or 2 which is multiplying moves to the right member dividing $x=3$

We check the solution

$2(3+5)+1=17$

$2\cdot 8+1=17$

$17=17$ So our solution is correct.

Exercise 11 Solve:

1. $x+7=10$
2. $x-3=21$
3. $t-6=7$
4. $x-22=13-4$
5. $4+y=54$
6. $12=3+z$
7. $m-24=17$
8. $1+p=2-89$
9. $x+x=37-2$
10. $23+y+2y=3$
11. $4x=29-1$

12. $100 = 16x + 20$

13. $5x + 3 = 20$

14. $10x = 13 - 2$

15. $2x + 7 = 6$

16. $2 = 7x - 3$

17. $12x - 7 = 13$

18. $3x - 5x = 23$
19. $5x + 7x = 23 - 1$

20. $44 + x = 12 - 3x$

Exercise 12 Solve:
1. $3(x - 2) = 7$

2. $2(x + 3) = 2$

3. $6(x - 5) = -2$

4. $7(x-1)+2x=20$

5. $3x-2(2x+3)=0$

6. $4(3x-5)-x=100$

7. $-21=6(3x-1)-13x$

8. $9(1-2x)+22x=1$

9. $2(3-x)+5x=1-3x$

10. $(3-x)\cdot 2-15x+2=3(1-3x)+2x-1$

Exercise 13 Express each problem as an equation and solve them

1. The sum of my age and 7 is 42. Find my age.

2. The difference of a number and 23 is 124. Find the number.

3. The quotient of 35 and a number is 7. Find the number.

4. If someone gives me 24€ I will have 34.23 € Find the money I have.

5. The double of my age minus 7 years is the age of my elder brother who is 19 years old.

6. The area of a square is 144 square metres. Find its length.

7. A teacher gives x coloured pencils each one of 8 girls except to one of them who only receives 4 pencils. The teacher gives 53 pencils in total. How many pencils did each girl receive?

8. Donald thinks of a number, multiplies it by 3 and subtracts 7. His answer is the double of the number plus 5 units. Which is the number?

9. In a triangle, the smallest angle is 20º less than the middle angle and the largest angle is twice the middle one. Find all the angles.

10. If we shorten 2 cm each of the two opposite sides of a square, we get a rectangle with an area which has 28 cm^2 less than the area of the square. Find out the perimeter of the rectangle.

Solutions

Exercise 1 1.1 $a+12$**, 1.2** $a-7$**, 1.3**$2a$**. Exercise 2 2.1** $x+10$**, 2.2** $123-x$**, 2.3** $2x$**, 2.4** $3x+3$**, 2.5** $\frac{x}{2}-7$**, 2.6** $\frac{3}{4}x+46$**. Exercise 3** a) $-\frac{1}{5}x^7$ is a monomial, $-\frac{1}{5}$ is the coefficient, x^7 is literal part, x is the variable; b) $2t^2$ is a monomial, 2 is the coefficient, t^2 is literal part, t is the variable; c) a + b is an algebraic expression but it is not a monomial; d) a^9 is a monomial, 1 is the coefficient, a^9 is literal part, a is the variable; e) $\sqrt{2}n^4$ is a monomial, $\sqrt{2}$ is the coefficient, n^4 is literal part, n is the variable; f) $3\sqrt{n}$ is an algebraic expression but it is not a monomial; g) $7abc^2$ is a monomial, 7 is the coefficient, abc^2 is literal part, a, b and c are the variables. **Exercise 4** a) $-5x$, b) $17x$, c) $7x$, d) $\frac{19}{2}x$, e) $21x^2$, f) $15x+2$, g) $-4a$, h) x^2+3x, i) $\frac{5}{3}x$. Exercise 5 a) $14x^2$, b) $30x^2$, c) $8x^2$, d) $15x^4$, e) $-46x^4$, f) $30x$.

Exercise 6 a) $-\frac{9}{5}$, b) 8, c) 14, d) -1, e) 90, f) $\frac{17}{2}$, g) 3. **Exercise 7** a) $15x$, b) x^2, c) 4ª, d) $21x^2$. **Exercise 8** a) $2x-8$, b) $12-6x$, c) $12x^2-6x$, d) $4t^2+16t$, e) $10x^2-15xy$, f) x^2-x, g) $14+14x$, h) $6x^2-2ax$. **Exercise 9** b) $A=14x-7$, c) $A=3x^2-6x$, d) $A=4x-28$. **Exercise 10** a) $5(2x+3)$, b) $4(x-3)$, c) $9(x-1)$, d) $x(3x-2)$, e) $3x(3x+1)$, f) $3(x-1)$, g) $3x(11x-1)$, h) $x(5-3x)$, g) It can not be factorized. **Exercise 11** 1. $x=3$, 2. $x=5$, 3. $t=13$, 4. $x=31$, 5. $y=50$, 6. $z=9$, 7. $m=41$, 8. $p=-88$, 9. $x=\frac{35}{2}$, 10. $y=-\frac{20}{3}$, 11. $x=4$, 12. $x=5$, 13. $x=\frac{17}{5}$, 14. $x=\frac{11}{10}$, 15. $x=-\frac{1}{2}$, 16. $x=\frac{5}{7}$, 17. $x=\frac{5}{3}$, 18. $x=-\frac{23}{2}$, 19. $x=\frac{11}{6}$, 20. $x=-8$.

Exercise 12 1. $x=\frac{13}{3}$, 2. $x=-2$, 3. $x=\frac{14}{3}$, 4. $x=3$, 5. $x=-6$, 6. $x=\frac{120}{11}$, 7. $x=-3$, 8. $x=-2$, 9. $x=-\frac{5}{6}$, 10. $x=\frac{3}{5}$.

Exercise 13 1. $x+7=42 \rightarrow x=35$; 2. $x-23=124 \rightarrow x=147$;
3. $\frac{35}{x}=7 \rightarrow x=5$; 4. $x+24=34.23 \rightarrow x=10.23$; 5. $2x-7=17 \rightarrow x=12$;
6. $x^2=144 \rightarrow x=12$; 7. $7x+4=53 \rightarrow x=7$; 8. $3x-7=2x+5 \rightarrow x=12$;
9. $x+x-20+2x=180 \rightarrow x=50$, the angles are 30º, 50º and 100º;
10. $x^2-28=x(x-4) \rightarrow x=7$, the perimeter is 20 cm.